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Stationary solutions to the nonlinear Schrödinger equation in the presence of third-order dispersion

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Abstract

We study stationary solutions of the nonlinear Schrödinger equation in the presence of small but non-zero third-order dispersion (TOD). Using a singular perturbation theory around the ideal soliton we calculate these solutions up to the second order in the TOD coefficient. The existence and linear stability of the stationary solutions is proved for any finite order of the perturbation theory. The results obtained by our numerical simulations of the nonlinear Schrödinger equation are in very good agreement with theory. The significance of these results for fibre optic communication systems is discussed.

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1. Introduction

Optical solitons have been the subject of extensive research efforts during the past decades due to their potential applications in long-distance communication systems and in optical interconnect technologies [1]. It is by now well established that the propagation of solitons in optical fibres can be described by the nonlinear Schrödinger equation (NLSE) [2, 3]. For ideal soliton propagation, the effects of refractive nonlinearity and second-order dispersion exactly balance each other. In this case, the soliton propagates without any changes in its parameters and without emitting any radiation, i.e., it is stationary. In practice, however, there exist physical processes, which lead to the breakdown of this stationary nature of soliton propagation. One of the processes of this type, which is of special importance for ultra-short pulses, is associated with third-order dispersion (TOD) [2, 4] . Indeed, numerical simulations have shown that solitons propagating in the presence of third-order dispersion emit continuous radiation, experience corruption of their shape, and undergo a shift in their group velocity [5]. These effects pose a serious limitation on the performance of ultra-short pulse optical fibre systems, and it is desirable to reduce them as much as possible.

Several authors studied the effects of TOD on pulse propagation at the zero dispersion wavelength [6–8]. In this paper we focus our attention on the more general problem, where the (dimensional) second-order dispersion coefficient is negative, and the fibre supports bright solitons. This problem was first considered by Kodama [9], who found that the TOD perturbation leads to a shift proportional to the TOD coefficient in the group velocity of the soliton. A more detailed analytical and numerical study of the dynamics of NLSE solitons in the presence of TOD was performed by Elgin [5, 10, 11]. Using the result of Kodama, Elgin employed an 'associate field' formalism to analyse the temporal and spectral properties of the radiation emitted by the soliton. Specifically, theoretical predictions for the velocities of the trailing and leading fronts of the emitted radiation were obtained and compared with the results of numerical simulations. However, some interesting and important questions regarding the existence of stationary pulses, the long-distance asymptotics of pulse propagation, and the dynamics of the soliton amplitude and phase, were not addressed.

Recently, the existence and stability of stationary solutions for the NLSE in the presence of small and non-zero TOD were assumed in [12, 13] for calculating the effects of TOD on two-soliton collisions. In these works the stationary solutions were calculated up to the first order in the TOD coefficient by using a singular perturbation theory around the ideal soliton developed by Kaup [14, 15]. One of the effects found in the first order is a change in frequency of the soliton, which is not accompanied by a respective change in the group velocity. The other effect was a time-dependent modulation of the phase. However, the existence and stability of the stationary solutions were not proved in [12, 13] beyond the first order. Moreover, even for the first order, no comparison with numerical simulations was presented. Therefore, the question of existence and stability of the stationary solutions, which is essential for obtaining the results presented in [12, 13] for the two-soliton collision problem, remains open.

In this paper we address this important question, and present an extensive and detailed investigation of stationary solutions in the presence of small non-zero TOD. We prove the existence and linear stability of the stationary solutions in any finite order of Kaup's perturbation theory. Furthermore, we calculate the stationary solutions up to the second order, which is essential for significantly suppressing undesirable variations in the soliton parameters. The predictions of our theory are in very good agreement with our numerical simulations of the NLSE with small but non-zero TOD. Finally, we discuss the possibility to obtain stationary solutions for other types of perturbations, and suggest that such stationary pulses can be used in optical fibre telecommunication systems to suppress radiation emission and other undesirable effects.

The rest of this paper is organized as follows. In section 2, the stationary solutions are calculated up to the second order in the TOD coefficient. In this section we also discuss the properties of these solutions and prove their existence in any finite order of the perturbation theory. In section 3, we present the results of our numerical simulations of the NLSE and compare them with our theoretical predictions. Section 4 is reserved for discussion. Appendix A contains the proof for the linear stability of the stationary solutions, and appendix B gives some auxiliary calculations related to Kaup's perturbation theory.

2. Perturbative calculation of the stationary solutions

Propagation of an electric field wave packet $\psi(t, z)$ through an optical fibre in the presence of small but non-zero TOD is described by the following modification of the nonlinear Schrödinger equation (see e.g., [2], p. 44)

$$\mathrm{i}\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = \mathrm{i}d_3\partial_t^3\psi. \tag{1}$$

Here z is the position along the fibre, t is the time in the retarded frame of reference, d_3 is the TOD coefficient, and the term $\mathrm{i}d_3\partial_t^3\psi$ accounts for the effect of TOD. Note that when

$d_3 \neq 0$, equation (1) is not integrable. However, in many practical cases $d_3 \ll 1$ [1], allowing a perturbative calculation about the integrable $d_3 = 0$ limit.

When $d_3 = 0$ the most general form of the fundamental soliton solution of equation (1) is given by [16]

$$\psi_{\rm sol}(t,z) = \eta \frac{\exp[\mathrm{i}\alpha + \mathrm{i}\beta(t-y) + \mathrm{i}(\eta^2-\beta^2)z]}{\cosh[\eta(t-y-2\beta z)]} \tag{2}$$

where η, β, α, and y are the soliton amplitude, phase velocity, phase and position, respectively. We call this solution the ideal soliton solution, and refer to equation (1) with $d_3 = 0$ as the ideal NLSE. We note that the group velocity of the ideal soliton is 2β.

Let us assume that $d_3 \ll 1$, and perturbatively obtain a stationary (z-independent) solution of equation (1). For simplicity, we assume that $\eta = 1$ and $\beta = \alpha = y = 0$. We emphasize, however, that the procedure described here can be easily generalized for any given values of these parameters [13]. Making the substitution

$$\psi(t,z) = \mathrm{e}^{\mathrm{i}z}\Psi(t) \tag{3}$$

into equation (1) we obtain an ordinary differential equation for $\Psi(t)$

$$\Psi'' - \Psi + 2|\Psi|^2\Psi = \mathrm{i}d_3\Psi'''. \tag{4}$$

Since $d_3 \ll 1$ we can look for solutions of equation (4) in the form of a perturbation series

$$\Psi(t) = \Psi_0(t) + \Psi_1(t) + \Psi_2(t) + \cdots \tag{5}$$

where $\Psi_0(t) = \cosh^{-1}(t)$ is of $O(1)$, Ψ_1 and Ψ_2 are the $O(d_3)$ and $O\left(d_3^2\right)$ terms, respectively, and ... stand for higher order terms.

Substituting (5) into equation (4) and keeping terms up to the first order in d_3, we arrive at

$$\hat{L}\begin{pmatrix}\Psi_1\\ {\Psi_1}^*\end{pmatrix} = \mathrm{i}d_3\Psi_0'''\begin{pmatrix}1\\1\end{pmatrix} \tag{6}$$

where the linear operator $\hat{L}$ is given by

$$\hat{L} = \left(\partial_t^2 - 1\right)\hat{\sigma}_3 + \frac{2}{\cosh^2(t)}(2\hat{\sigma}_3 + \mathrm{i}\hat{\sigma}_2) \tag{7}$$

and $\hat{\sigma}_2$ and $\hat{\sigma}_3$ stand for the Pauli matrices. The operator $\hat{L}$ describes the evolution of linear perturbations around the soliton solution (2) of the ideal NLSE. The complete set of the eigenfunctions of $\hat{L}$, which was obtained by Kaup [14, 15], include an infinite (continuous) set of unlocalized modes f_k and $\bar{f}_k$, obeying

$$\hat{L}f_k = (k^2+1)f_k \qquad \hat{L}\bar{f}_k = -(k^2+1)\bar{f}_k. \tag{8}$$

There are also four discrete (localized) modes, f_0, f_1, f_2 and f_3, obeying

$$\begin{aligned} \hat{L}f_0 &= 0 & \hat{L}f_1 &= 0 \\ \hat{L}f_2 &= -2f_1 & \hat{L}f_3 &= -2f_0. \end{aligned} \tag{9}$$

The explicit expressions for these eigenmodes are given in appendix B. In this appendix we also show that small (infinitesimal) changes in the soliton parameters α, y, β and η can be expressed in terms of the four localized eigenmodes f_0, f_1, f_2 and f_3, respectively.

It is useful to expand $\tilde{\Psi}_1$ in a series of the eigenmodes of $\hat{L}$

$$\begin{pmatrix}\Psi_1\\ \Psi_1^*\end{pmatrix} = \tilde{c}_0^1 f_0(t) + \tilde{c}_1^1 f_1(t) + \tilde{c}_2^1 f_2(t) + \tilde{c}_3^1 f_3(t) + \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\left[c_k^1 f_k(t) + c_k^{1*}\bar{f}_k(t)\right] \tag{10}$$

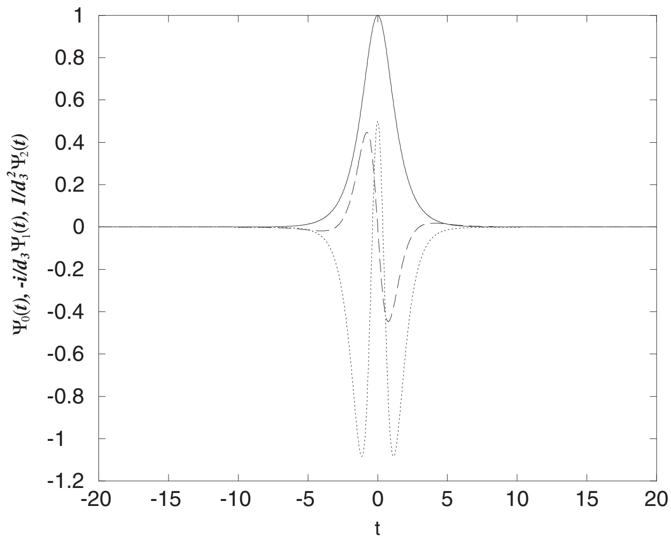

Figure 1. The time dependence of the zero, first- and second-order terms Ψ_0, Ψ_1 and Ψ_2, appearing in the perturbative expansion (5) for the stationary solution. The solid line represents $\Psi_0(t)$, the dashed line stands for $-\mathrm{i}\Psi_1(t)/d_3$, and the dotted line for $\Psi_2(t)/d_3^2$.

where subscripts denote the corresponding eigenmodes, and the superscript 1 denotes the first order with respect to d_3. The expansion of the right-hand side of equation (6) over these eigenmodes is

$$\mathrm{i}d_3\Psi_0'''(t)\begin{pmatrix}1\\1\end{pmatrix} = \mathrm{i}d_3 f_1(t) + \frac{d_3}{4}\int_{-\infty}^{+\infty}\mathrm{d}k\left[\frac{k(k+\mathrm{i})^2 f_k(t)}{\cosh(\pi k/2)} - \frac{k(k-\mathrm{i})^2 \bar{f}_k(t)}{\cosh(\pi k/2)}\right]. \tag{11}$$

Substituting equations (10) and (11) into equation (6), and using relations (8) and (9) we obtain the following expressions for the expansion coefficients:

$$\tilde{c}_2^1 = -\frac{\mathrm{i}d_3}{2} \qquad \tilde{c}_3^1 = 0 \tag{12}$$

and

$$c_k^1 = \frac{\pi d_3 k(k+\mathrm{i})}{2(k-\mathrm{i})\cosh(\pi k/2)}. \tag{13}$$

The expression for Ψ_1 can be further simplified by using the explicit expressions for the eigenmodes of $\hat{L}$. This calculation yields

$$\begin{pmatrix}\Psi_1\\ \Psi_1^*\end{pmatrix} = \tilde{c}_0^1 f_0(t) + \tilde{c}_1^1 f_1(t) - \frac{\mathrm{i}d_3 t}{2\cosh(t)}\begin{pmatrix}1\\-1\end{pmatrix} - \frac{\mathrm{i}d_3}{2}I(t)\begin{pmatrix}1\\-1\end{pmatrix} \tag{14}$$

where the function $I(t)$ is given by

$$I(t) = \int_{-\infty}^{+\infty}\mathrm{d}k\,\frac{k\left[k\cos(kt)\tanh(t) + \frac{1}{2}(k^2-1)\sin(kt)\right]}{(k^2+1)\cosh(\pi k/2)}. \tag{15}$$

The function $\Psi_1(t)$ is shown in figure 1.

Let us discuss the properties of the first-order term $\Psi_1(t)$. The coefficients $\tilde{c}_0^1$ and $\tilde{c}_1^1$ in equation (14) correspond to $O(d_3)$ corrections to the soliton phase and position, respectively (see appendix B). We note that these terms remain arbitrary, which is a direct result of the fact that the eigenvalues of f_0 and f_1 are zero. This means that there is actually a *family* of stationary solutions, which depend on the parameter d_3 (and in the general case also on the four soliton parameters η, β, α and y). The values of the coefficients $\tilde{c}_0^1$ and $\tilde{c}_1^1$, which can be fixed by the initial condition at $z = 0$, determine which member of the family of stationary solutions is selected. For simplicity we choose $\tilde{c}_0^1 = \tilde{c}_1^1 = 0$ and obtain that $\Psi_1(t)$ is odd in time and purely imaginary. The third term on the right-hand side of equation (14) corresponds to an $O(d_3)$ frequency change, which is not accompanied by a corresponding change in the group velocity (see appendix B). This term is responsible for the elimination of the $O(d_3)$ shift in the group velocity and the corresponding secular growth in the soliton position, which are observed in the dynamical problem. The fourth term corresponds to an $O(d_3)$ time-dependent modulation of the phase. Although this term is contributed by the continuous spectrum eigenmodes, it is localized, i.e., it decays exponentially in t for $t \gg 1$. Indeed, using the Residue Theorem one can show that for large t, $\Psi_1(t)$ is given by

$$\Psi_1(t) = \mathrm{i}d_3[(t-3)\,\mathrm{e}^{-t} + O(\mathrm{e}^{-3t})]. \tag{16}$$

Since in obtaining the perturbative expansion (5) we assumed that $|\Psi_0| \gg |\Psi_1|$, and since for large values of t $|\Psi_0| \sim \mathrm{e}^{-t}$ and $|\Psi_1| \sim d_3 t\,\mathrm{e}^{-t}$, the solution (14) is valid only for $t \ll 1/d_3$. However, this limitation is not important in practice since for $d_3 \ll 1$ both Ψ_0 and Ψ_1 are exponentially small already at $1 \ll t \ll 1/d_3$.

Let us now briefly discuss the calculation of the second-order term $\Psi_2(t)$. The linearized form of equation (4) in this order can be written in the form

$$\hat{L}\begin{pmatrix}\Psi_2\\ {\Psi_2}^*\end{pmatrix} = [\mathrm{i}d_3\Psi_1''' - 2\Psi_0|\Psi_1|^2]\begin{pmatrix}1\\ -1\end{pmatrix}. \tag{17}$$

We solve this equation by the same method used in the first order. That is, we first expand Ψ_2 and the right-hand side of equation (17) in terms of a series of the eigenmodes of $\hat{L}$. The expansion of Ψ_2 is

$$\begin{pmatrix}\Psi_2\\ \Psi_2^*\end{pmatrix} = \tilde{c}_0^2 f_0(t) + \tilde{c}_1^2 f_1(t) + \tilde{c}_2^2 f_2(t) + \tilde{c}_3^2 f_3(t) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\left[c_k^2 f_k(t) + c_k^{2*}\bar{f}_k(t)\right] \tag{18}$$

where the superscript 2 denotes second order with respect to d_3. Substituting (18) into equation (17) and using relations (8) and (9), we obtain

$$\tilde{c}_2^2 = 0 \qquad \tilde{c}_3^2 = 1.3306168 d_3^2 \tag{19}$$

while the coefficients $\tilde{c}_0^2$ and $\tilde{c}_1^2$ remain arbitrary. The values of the coefficients c_k^2 are shown in figure 2.

Figure 1 shows the second-order term $\Psi_2(t)$ for $\tilde{c}_0^2 = \tilde{c}_1^2 = 0$. We note that this term is even with respect to t and real. Hence, it follows that the absolute value of the stationary solution $|\Psi| = |\Psi_0 + \Psi_1 + \Psi_2|$, calculated up to the second order in d_3, is symmetric with respect to t. This is in contrast to the behaviour observed in the dynamical problem, where the ideal soliton is taken as the initial condition. In this case the pulse acquires an asymmetric form [5]. Indeed, a simple linear analysis shows that the asymmetric form of the pulse in the dynamical problem is a direct result of the change in the pulse position, which is proportional to $d_3 z$. For the stationary solution found here, the position does not change with z, and as a result, the shape remains symmetric. It was also checked that for $t \gg 1$ this term decays like $d_3^2 t^2\,\mathrm{e}^{-t}$. Hence, the condition for the validity of the second order of the perturbation theory is the same as for the first order, i.e., $t \ll 1/d_3$.

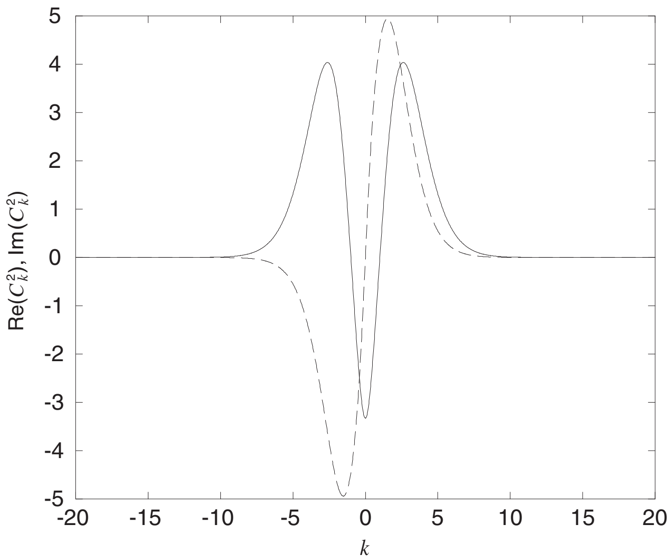

Figure 2. The real part (solid) and the imaginary part (dashed) of the coefficients c_k^2, appearing in expression (18).

Consider now the mth order of the perturbation theory. In this order the linearized form of equation (4) becomes

$$\Psi_m'' - \Psi_m + 2\Psi_0^2[2\Psi_m + \Psi_m^*] = \mathrm{i}d_3\Psi_{m-1}''' - 2\sum_{i,j,k}\Psi_i\Psi_j\Psi_k^*\delta_{i+j+k-m} \tag{20}$$

where δ stands for the Kronecker delta function. In writing equation (20) it is understood that the terms $4\Psi_0^2\Psi_m$ and $2\Psi_0^2\Psi_m^*$ are not included in the triple sum appearing on the right-hand side of the equation. Note the following property of equation (20). The projection of the left-hand side of this equation on the eigenmodes f_2 and f_3 are zero for any m. However, when the right-hand side of equation (20) is either imaginary and even or real and odd, it has projections on these eigenmodes. In this case equation (20) is not valid, and the perturbation scheme breaks down. This would correspond to the situation where in the dynamical problem the perturbation leads to a linear dependence on z of either the soliton amplitude or its frequency. However, we now show that in the current problem this is not the case, and in fact, equation (20) does have a solution in any order m of the perturbation theory.

For simplicity, we first assume that the coefficients $\tilde{c}_0^i = \tilde{c}_1^i$ are zero in any order $i < m$. As we have shown earlier, Ψ_1 is odd and imaginary, while Ψ_0 and Ψ_2 are even and real. Let us assume by induction that Ψ_i is even and real for any even $i < m$, and odd and imaginary for any odd $i < m$. Consider first the case where m is even. In this case Ψ_{m-1} is odd and pure imaginary, and as a result, $\mathrm{i}d_3\Psi_{m-1}'''$ is real and even. Hence, its projections on the eigenmodes f_2 and f_3 are zero. Next we note that all terms in the sum $\sum_{i,j,k}\Psi_i\Psi_j\Psi_k^*\delta_{i+j+k-m}$ are even and real. Therefore, the projections of this term on the eigenmodes f_2 and f_3 are zero as well. It follows that the stationary solution Ψ_m does exist. The proof for the case where m is odd can be done in a similar manner. In addition, this result can be generalized for the case where the coefficients $\tilde{c}_0^i$ and $\tilde{c}_1^i$ are non-zero. Indeed, $\tilde{c}_0^i f_0(t)$ is even and imaginary, and $\tilde{c}_1^i f_1(t)$ is odd and real. Therefore, in this case for any $i < m$, Ψ_i is a sum of terms, which are either even and real or odd and imaginary. It follows that the right-hand side of equation (20) is of the same form, and thus, its projections on f_2 and f_3 are zero. We conclude that the stationary

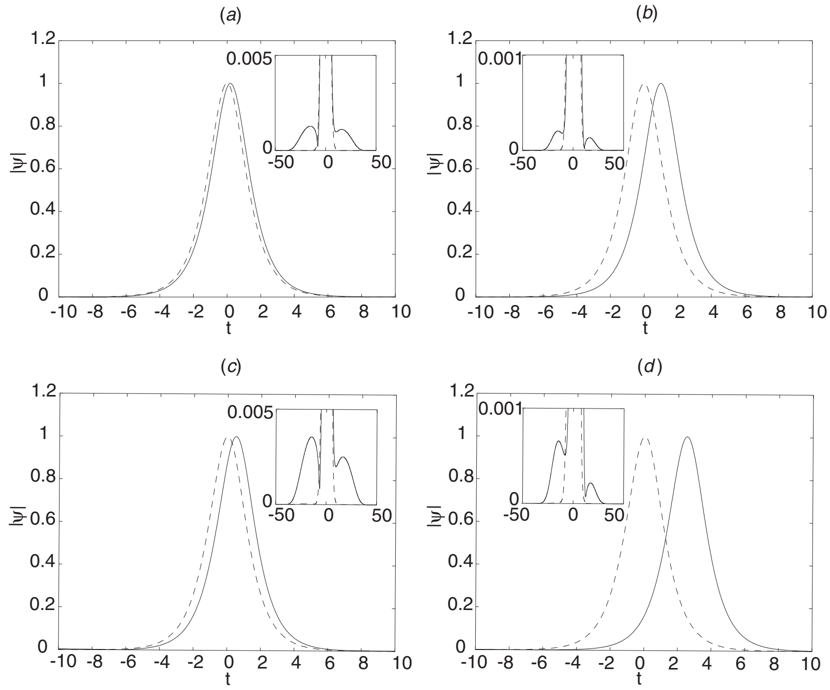

Figure 3. Absolute value of the optical field $|\psi(t, z)|$ as a function of t for an initial condition in the form of the stationary pulse (5) (dashed), and for a soliton initial condition (solid). The upper row shows $|\psi(t, z)|$ for $d_3 = 0.02$ at $z = 10$ (a) and at $z = 50$ (b), while the lower row shows $|\psi(t, z)|$ for $d_3 = 0.05$ at $z = 10$ (c) and at $z = 50$ (d). The insets show the same data for a wider range of t and a smaller range of $|\psi|$.

solution exists in any order m of the perturbation theory. The proof for its linear stability is given in appendix A.

3. Numerical simulations

In order to confirm the predictions of our theory we performed numerical experiments with the NLSE in the presence of small but non-zero TOD (equation (1)). Two types of initial conditions were considered: the stationary solution (5), calculated up to the second order in d_3 and an initial condition in the form of an ideal soliton, $\psi(t, z = 0) = \cosh^{-1}(t)$.

We choose to utilize the split-step method with the periodic boundary conditions, which has been used extensively in the pulse propagation problems in nonlinear optics. For the detailed description of this technique, we refer [2, 3]. As we remarked previously, solitons emit radiation in the presence of TOD. This radiation moves away from the solitons and eventually interact with the artificial boundaries in a finite computational domain, which causes severe computational errors. In order to overcome these numerical artifacts, we apply an artificial damping at the vicinity of edges to suppress the radiation at this region. It is also worth mentioning that the size of the computational domain needs to be large enough so that these absorbing layers do not interfere the internal solutions. Here, we take the computational boundary $-L \leqslant t \leqslant L$ with $L = 200$.

The dynamical evolution of the optical field $\psi(t, z)$ under equation (1) is illustrated in figure 3, for the two types of initial conditions, and for two different values of d_3: $d_3 = 0.02$

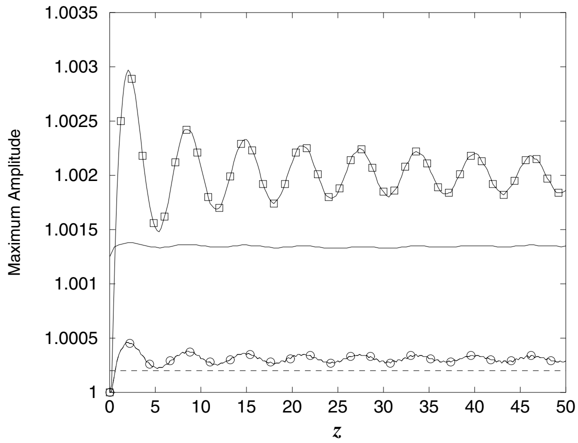

Figure 4. The pulse amplitude, defined as the value of $|\psi(t,z)|$ at its maximum, as a function of z. The dashed and solid lines correspond to an initial condition in the form of the stationary solution for $d_3 = 0.02$ and $d_3 = 0.05$, respectively. The circles and squares correspond to a soliton initial condition with $d_3 = 0.02$ and $d_3 = 0.05$, respectively.

and $d_3 = 0.05$. It is shown that for both values of d_3 the position of the stationary pulse remains approximately unchanged, whereas the soliton position changes significantly with z. In addition, as can be seen from the insets of figure 3, the shape of the stationary solution remains approximately unchanged, while the soliton develops the well-known asymmetric radiative tail. We note that the radiative tail of the soliton decreases with increasing distance along the fibre. It should also be noted that for $d_3 = 0.05$ a careful examination of the data shows that the stationary solution does develop an asymmetric radiative tail, which also decreases with z. However, this tail is of order d_3^4, i.e., it is a result of the fact that the stationary solution was calculated only up to the second order in d_3. For $d_3 = 0.02$ this tail is too small to be captured in the simulation.

The evolution of the pulse amplitude with distance z along the fibre is presented in figure 4. Here, we remark that the pulse amplitude is defined as the value of $|\psi(t,z)|$ at its maximum, and in general it is different from the ideal soliton amplitude η. One can see that the amplitude of the stationary pulse remains constant up to $O\left(d_3^4\right)$ corrections, resulting from the fact that the stationary solution was calculated only up to $O\left(d_3^2\right)$. It is also shown that the soliton amplitude oscillates, and the amplitude of these oscillations is proportional to d_3^2 and decreases with z. Note that since Ψ_1 in equation (14) is pure imaginary, one has to calculate Ψ_2 in order to suppress the oscillations of the stationary pulse amplitude.

Figure 5 shows the position of the pulse y as a function of distance along the fibre z for the two types of initial conditions and for the values of $d_3 = 0.02$ and 0.05, respectively. Here, we use the centre of the mass as the numerical definition of the pulse position. That is, we calculate $\int \mathrm{d}t[|\psi(t,z)|t/\int \mathrm{d}t|\psi(t,z)|]$ and the integrations are performed in the vicinity of the point where $|\psi(t,z)|$ attains its maximum. One can see that the position of the stationary pulse remains approximately unchanged, while the soliton position grows like $\sim d_3 z$. This latter behaviour corresponds to a shift proportional to d_3 in the soliton group velocity [5, 9]. A more careful numerical study of $y(z)$ for the stationary pulse dynamics shows that it grows like $\sim d_3^3 z$. A similar analysis for the pulse phase α is shown in figure 6. While the soliton phase grows with z like $d_3^2 z$, the phase of the stationary pulse remains constant up to corrections of

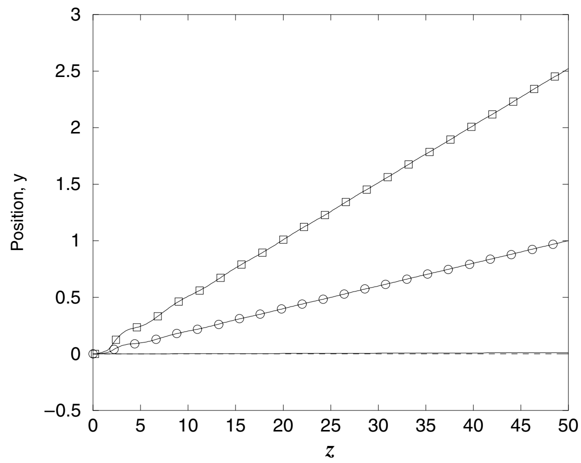

Figure 5. Position of the pulse y as a function of distance along the fibre z. The dashed and solid lines correspond to an initial condition in the form of the stationary solution for $d_3 = 0.02$ and $d_3 = 0.05$, respectively. The circles and squares correspond to a soliton initial condition with $d_3 = 0.02$ and $d_3 = 0.05$, respectively.

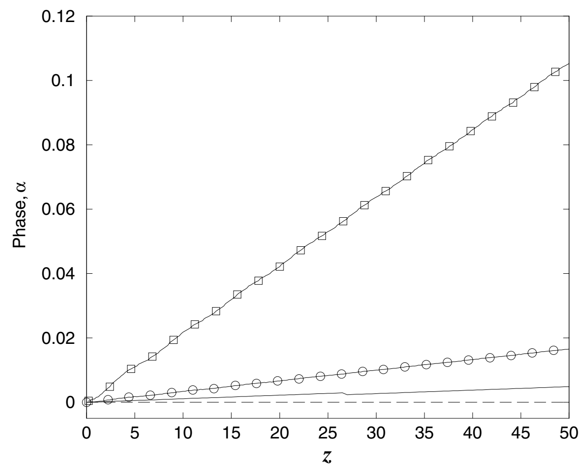

Figure 6. The pulse phase α as a function of distance along the fibre z. The dashed and solid lines correspond to an initial condition in the form of the stationary solution for $d_3 = 0.02$ and $d_3 = 0.05$, respectively. The circles and squares correspond to a soliton initial condition with $d_3 = 0.02$ and $d_3 = 0.05$, respectively.

order $d_3^3 z$ or higher. Again we note that to obtain such a suppression of the change of phase, one has to calculate the stationary solution up to the second order in d_3.

We conclude this section by pointing out that our numerical simulations also test the linear stability of the stationary pulses. Indeed, since the initial stationary pulses were calculated only up to the second order in d_3, the evolution of the pulses is in fact subject to a perturbation of order d_3^3. The fact that no instability is observed in the simulations is in agreement with our prediction for linear stability of these pulses.

4. Discussion

We investigated stationary solutions of the nonlinear Schrödinger equation in the presence of small but non-zero third-order dispersion. We calculated these solutions up to the second order in the third-order dispersion coefficient by using a singular perturbation theory around the ideal soliton. Moreover, we proved the existence and linear stability of the stationary solutions for any finite order of the perturbation theory. The results obtained by our numerical simulations of the nonlinear Schrödinger equation are found to be in very good agreement with the predictions of our theory.

Let us discuss the implications of our results for a typical short pulse optical fibre system. The relation between the pulse width τ_0, the dimensional second- and third-order dispersion coefficients β_2 and β_3, respectively, and the dimensionless TOD coefficient d_3 is $d_3 = -\beta_3/(3\beta_2\tau_0)$. For $\tau_0 = 0.4$ ps and $\beta_2 = -2\ \mathrm{ps}^2\ \mathrm{km}^{-1}$, $\beta_3 = 0.12\ \mathrm{ps}^3\ \mathrm{km}^{-1}$ corresponds to $d_3 = 0.05$, and $\beta_3 = 0.048\ \mathrm{ps}^3\ \mathrm{km}^{-1}$ corresponds to $d_3 = 0.02$. As we have seen in the previous section, by launching a pulse which is stationary up to the second order in d_3 (instead of an ideal soliton), it is possible to reduce the effects of amplitude oscillations and z-dependent position shift by a factor of $d_3^{-2} = 400$ and $d_3^{-2} = 2500$ for the two short pulse systems, respectively. Launching such a stationary pulse also results in a reduction in the z-dependent phase shift by at least a factor of $d_3^{-1} = 20$ and $d_3^{-1} = 50$ for the aforementioned systems. It was also found in [12, 13] that the intensity of radiation emitted as a result of a collision between two such stationary pulses, belonging to two different frequency channels, is proportional to $d_3^2(\Delta\beta)^{-4}$, where $\Delta\beta$ is the frequency difference between the two channels. Since in real optical fibre systems we find $\Delta\beta^{-1} \ll 1$, this latter result means that the intensity of the collision induced radiation is of sixth combined order in the two small parameters d_3 and $\Delta\beta^{-1}$, i.e., it is very small.

We note that the results obtained in this work are in accordance with the robustness conjecture by Menyuk [17]. According to this conjecture, Hamiltonian deformations of the NLSE, such as third-order dispersion, lead to radiation emission, corruption of shape, change in the soliton parameters, but not to the complete destruction of the solitons. However, this robustness conjecture was justified based on two additional hypotheses, and not derived from first principles. In this study we were able to explicitly calculate the form of the stationary solutions with account of TOD, to prove the existence and stability of these solutions, and to confirm these predictions by numerical simulations.

We should also point out that some authors suggested to identify the radiation emitted by solitons propagating in the presence of TOD as Cherenkov radiation [18]. However, it is known that when the soliton velocity is smaller than the minimum phase velocity of the linear waves in the medium, or larger than the maximum phase velocity of these linear waves, the condition for emission of Cherenkov radiation is not satisfied, and solitons can exist [19]. From equation (8) it follows that in the current problem the phase velocity of the linear waves will be given by $(k^2 + 1)/k$, while the TOD induced velocity is proportional to d_3. Hence, for sufficiently small d_3 the condition for emission of Cherenkov radiation is not satisfied and solitons can exist. Moreover, the $O(1)$ gap between the eigenvalues of the continuous spectrum eigenmodes and the four localized eigenmodes (cf equation (8) with equation (9)) guarantees that this result remains valid for the stationary solutions (5) in any finite order of the perturbation theory, provided d_3 is small enough.

It was mentioned in section 3 that for the case of soliton propagation, the TOD induced radiative tail and amplitude oscillations decrease with increasing z. Therefore, an interesting question concerns the asymptotic form of $|\psi(t, z)|$ for $z \gg 1$, and its relation to the stationary solution reported here. It is also worth mentioning that the third-order dispersion is not the only effect which breaks the stationary nature of soliton propagation in optical fibres. Indeed, other destructive effects such as Raman scattering and self-steepening can also lead to emission of radiation, corruption of the pulse shape and change of the soliton parameters. It will be interesting to see if one can find stationary pulses in the presence of such perturbations. We hope that this study will motivate experimental efforts to realize such stationary pulses.

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Appendix A. Linear stability of the stationary solution

In this appendix we prove the linear stability of the stationary solution in any finite order m of the perturbation theory. For this aim we consider a small perturbation on top of the stationary solution, calculated up to order d_3^m

$$\Psi(t;m) = \Psi_0(t) + \Psi_1(t) + \cdots + \Psi_m(t). \tag{A1}$$

We substitute into equation (2) a solution of the form

$$\tilde{\Psi}(t,z) = [\Psi(t;m) + v(t,z)]\exp(\mathrm{i}z + \mathrm{i}\alpha) \tag{A2}$$

where, $|v(t,z)| \ll |\Psi_m(t)|$. Keeping only linear terms in v we obtain

$$\mathrm{i}\partial_z v + \left[\partial_t^2 v - v + 2\Psi_0^2(2v + v^*)\right] - \mathrm{i}d_3\partial_t^3 v + 4\sum_{i,j}\Psi_i\Psi_j^* v + 2\sum_{i,j}\Psi_i\Psi_j v^* = 0. \tag{A3}$$

In writing equation (A3) it is understood that the terms $4\Psi_0^2 v$ and $2\Psi_0^2 v^*$ are not included in the double sums appearing on the left hand side of the equation. Combining this equation with the corresponding equation for v^* we arrive at

$$\mathrm{i}\partial_z\begin{pmatrix} v \\ v^* \end{pmatrix} + \hat{L}_{\mathrm{tot}}\begin{pmatrix} v \\ v^* \end{pmatrix} = 0, \tag{A4}$$

where $\hat{L}_{\mathrm{tot}}$ is defined by

$$\hat{L}_{\mathrm{tot}} = \hat{L} - \mathrm{i}d_3\hat{I}\partial_t^3 + \sum_{l=1}^{m}\hat{L}_l \tag{A5}$$

and $\hat{I}$ is the unit matrix. Assuming that $\tilde{c}_0^l = \tilde{c}_1^l = 0$ for any $l \leqslant m$, the operators $\hat{L}_l$ appearing in equation (A5) are given by

$$\hat{L}_l = 4\left(|\Psi_{l/2}|^2 + \sum_{i\neq j}\Psi_i\Psi_j^*\delta_{i+j-l}\right)\hat{\sigma}_3 + 2\mathrm{i}\left(\Psi_{l/2}^2 + \sum_{i\neq j}\Psi_i\Psi_j\delta_{i+j-l}\right)\hat{\sigma}_2 \quad \text{for even } l \tag{A6}$$

and

$$\hat{L}_l = 4\left(\sum_{i\neq j}\Psi_i\Psi_j\delta_{i+j-l}\right)\hat{\sigma}_1 \quad \text{for odd } l. \tag{A7}$$

It is now straightforward to show that the operator $\hat{L}_{\mathrm{tot}}$ satisfies the relation

$$\hat{\sigma}_1\hat{L}_{\mathrm{tot}}\hat{\sigma}_1 = -L_{\mathrm{tot}}^* \tag{A8}$$

from which it follows that its eigenvalues are real. Thus, the stationary solution is linearly stable in any finite order m of the perturbation theory. The generalization of this proof for the case where the coefficients $\tilde{c}_0^l$ and $\tilde{c}_1^l$ are non-zero is straightforward.

Appendix B. The eigenmodes of the operator $\hat{L}$

In this appendix we give the explicit formulas for the eigenmodes of the linear operator $\hat{L}$, as found by Kaup [14, 15]. We also show that small (infinitesimal) changes in the four parameters of the soliton can be expressed in terms of the four discrete eigenmodes of this operator. We then explain how we use these four expressions to identify terms on the right-hand side of equation (14) with small corrections to the soliton parameters.

The four localized eigenmodes of $\hat{L}$ are given by

$$f_0(t) = \frac{1}{\cosh(t)}\begin{pmatrix}1\\-1\end{pmatrix} \tag{B1}$$

$$f_1(t) = \frac{\tanh(t)}{\cosh(t)}\begin{pmatrix}1\\1\end{pmatrix} \tag{B2}$$

$$f_2(t) = \frac{t}{\cosh(t)}\begin{pmatrix}1\\-1\end{pmatrix} \tag{B3}$$

and

$$f_3(t) = \frac{t\tanh(t)-1}{\cosh(t)}\begin{pmatrix}1\\1\end{pmatrix}. \tag{B4}$$

The unlocalized eigenmodes of $\hat{L}$ are given by

$$f_k = \exp[\mathrm{i}kt]\left\{1 - \frac{2\mathrm{i}k\exp[-t]}{(k+\mathrm{i})^2\cosh[t]}\right\}\begin{pmatrix}0\\1\end{pmatrix} + \frac{\exp[\mathrm{i}kt]}{(k+\mathrm{i})^2\cosh^2[t]}\begin{pmatrix}1\\1\end{pmatrix} \tag{B5}$$

and

$$\bar{f}_k = \exp[-\mathrm{i}kt]\left\{1 + \frac{2\mathrm{i}k\exp[-t]}{(k-\mathrm{i})^2\cosh[t]}\right\}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\exp[-\mathrm{i}kt]}{(k-\mathrm{i})^2\cosh^2[t]}\begin{pmatrix}1\\1\end{pmatrix} \tag{B6}$$

where k runs from $-\infty$ to $+\infty$.

Consider the general form of the fundamental soliton solution of the ideal NLSE with $\beta = 0$ (see equation (2))

$$\psi_{\mathrm{sol}} = \frac{\eta\exp(\mathrm{i}\alpha + \mathrm{i}\eta^2 z)}{\cosh(x)} \tag{B7}$$

where $x = \eta(t-y)$. Let us denote

$$\tilde{\psi}_{\mathrm{sol}} = \frac{\eta}{\cosh(x)} \tag{B8}$$

and calculate the infinitesimal changes $\delta\tilde{\psi}_{\mathrm{sol}}$ originating from infinitesimal changes in α, y, β and η. For $\delta\alpha \ll 1$ and $\delta y \ll 1$ we obtain

$$\begin{pmatrix}\delta\tilde{\psi}_{\mathrm{sol}}\\ \delta\tilde{\psi}^*_{\mathrm{sol}}\end{pmatrix}_{\delta\alpha} = \mathrm{i}\eta\delta\alpha f_0(x) \tag{B9}$$

and

$$\begin{pmatrix}\delta\tilde{\psi}_{\mathrm{sol}}\\ \delta\tilde{\psi}^*_{\mathrm{sol}}\end{pmatrix}_{\delta y} = \eta^2\delta y f_1(x) \tag{B10}$$

respectively. From equations (B9) and (B10) it follows that the eigenmodes f_0 and f_1 are associated with small changes in α and y, respectively. For $\delta\beta \ll 1$ we obtain

$$\begin{pmatrix}\delta\tilde{\psi}_{\mathrm{sol}}\\ \delta\tilde{\psi}^*_{\mathrm{sol}}\end{pmatrix}_{\delta\beta} = \mathrm{i}\delta\beta f_2(x) + 2\eta z\delta\beta f_1(x). \tag{B11}$$

The first term on the right-hand side of equation (B11) originates from a small change in β in the argument of the exponential factor in (B7). Therefore, it corresponds to a change in the soliton frequency. The second term on the right-hand side of equation (B11) comes from a small change in β in the variable x defined above, and corresponds to a change in the group velocity of the soliton. For $\delta\eta \ll 1$ we obtain

$$\begin{pmatrix} \delta\tilde{\psi}_{\mathrm{sol}} \\ \delta\tilde{\psi}^*_{\mathrm{sol}} \end{pmatrix}_{\delta\eta} = -\delta\eta f_3(x) + 2\mathrm{i}\eta^2 z\delta\eta f_0(x). \tag{B12}$$

The first term on the right-hand side of this equation corresponds to a change in the soliton amplitude. The second term, which originates from a small change in η in the argument of the exponential factor in (B7), corresponds to a z dependent change in the soliton phase.

We can now use relations (B9)–(B12) to identify terms appearing on the right-hand side of equation (14) with $O(d_3)$ corrections to the soliton parameters. For this aim we compare the right-hand side of equation (14) with the right-hand sides of relations (B9)–(B12). We note that, while relations (B9)–(B12) describe the z dependent evolution of an ideal soliton under some perturbation, the terms on the right-hand side of equation (14) are independent of z by construction. In other words, equation (14) does not contain any z dependent terms such as $2\eta z\delta\beta f_1(x)$ or $2\mathrm{i}\eta^2 z\delta\eta f_0(x)$.

The first term on the right-hand side of equation (14) has the same form as the term on the right-hand side of equation (B9). We therefore identify the coefficient $\tilde{c}_0^1$ as corresponding to an $O(d_3)$ correction to the soliton phase. In the same manner, by comparing the second-order term on the right-hand side of equation (14) with the term on the right-hand side of equation (B10) we conclude that the coefficient $\tilde{c}_1^1$ corresponds to an $O(d_3)$ correction to the soliton position. The third term on the right-hand side of equation (14) has the same form as the first term on the right-hand side of equation (B11). Thus, we identify this term as an $O(d_3)$ correction to the soliton frequency. Since equation (14) does not contain any z-dependent terms, this correction to the frequency is not accompanied by any $O(d_3)$ change in the group velocity.

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